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On the possibility of turbulent thickening of weak shock waves

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This paper examines the possible thickening of an initially sharp sonic boom by the turbulence it encounters in passing to the ground. Three apparently different viewpoints, all indicating substantial thickening, are shown to be actually identical and to give an irrelevant upper bound on wave thickness. All three approaches describe only the apparent mean diffusion induced by random convection of a sharp wave about its nominal position. Although a wave-front folding mechanism ultimately accounts for an apparent thickening as individual rays are weakened and tangled by turbulence, this process is too slow to be effective in the practical boom situation. The paper then considers what linear thickening of a wave packet results from propagation through atmospheric turbulence and concludes that, in the relevant limit, a wave may be thickened by a factor of about 2 at the most. The conclusion is therefore reached that atmospheric turbulence cannot be the cause of the thousandfold discrepancy between the measured wave fronts and their Taylor thickness.

1. Introduction

The steady waves generated by an aircraft flying supersonically through a quiescent atmosphere develop at large distances into a characteristic 'N' structure. The leading and trailing shocks are separated by a gradual expansion, which takes place over a distance of some 50 metres. Experiment (figure 1 (*a*), plate 1) confirms Whitham's (1956) predication of this asymptotic wave form. The compressive waves, on the other hand, (figure 1 (*b*)) are of thicknesses one thousand times greater than their Taylor (1910) values, so that they are governed by mechanisms different from the balance between nonlinear steepening and molecular diffusion. The origin of this abnormality has been a matter of considerable debate. It is true that the wave forms display irregularities of fine detail which vary in successive realizations and that there is difficulty in drawing firm conclusions from any individual case. (Figures 1 (*a*), 2 (*a*) and 3 (*a*) (plate 1) are all recordings taken under nominally identical conditions.) However, there is little doubt that the compression proceeds at what is by normal shock standards an extremely sedate pace and that there are pronounced irregularities (spikes) which distinguish the real event from the simple model. That model evidently omits significant effects of the real atmosphere and it is both scientifically and technologically important to discover what those effects might be. The practical significance arises because the fine structure of a sonic boom seems to influence its power of 'annoyance' (Rice & Lilley 1969; Rice 1972).

The irregular spikey structure is now known to originate in the interaction of the primary wave with the turbulence it encounters while propagating through the atmosphere. Secondary waves are produced by the interaction, and these waves are well defined by scattering theory which Crow (1969) used to explain the detailed spike structure. It is natural to seek also the cause of 'shock thickening' in the same effect. Indeed this has been done by several authors though they fail to convey the same conviction on this point as does Crow's explanation of 'spikes'.

The subject was really anticipated by Lighthill (1953) when he showed that waves would be scattered by turbulence and that the energy of an incident wave is attenuated by an amount comparable with that in the scattered field. The suggestion that turbulence causes an initially sharp-fronted wave to shed its high frequency energy into sharp spikes then seems irresistable. But Lighthill warned that this view should be taken with extreme caution because the energy arguments were in the *stochastic* mean, and might only be approximated to in a single realization.

The most determined attempt to relate sonic-boom shock structure to atmospheric turbulence was made by Plotkin & George (1972). Their perturbation scheme steered clear of obvious interpretational difficulties that arise in translating the consequences of a statistical theory to an individual realization. They took care not to concentrate, as Howe (1971*a*) had done when considering waves on a random string, on properties established in the stochastic mean. In fact they were aware that such *mean* wave fields lost 'energy' without consequent rounding of any sharp individual wave. Diffuse mean wave profiles are inevitably formed when sharp wave fronts execute random walks about some mean position, and in that case it is clearly wrong to associate 'mean-square' properties with energy, and to invoke energy conservation arguments to demand that the incident wave supply the energy of the scattered field. The mean wave analysis therefore seems impotent in the search for a wave thickening mechanism, though it would be relevant if it could be established that wave fronts cannot execute random walks. That seems to us unlikely.

There appear then to be three distinct approaches to the turbulent origin of wave thickening.

(i) A scheme based on Lighthill's (1953) scattering theory with an assumed energy conservation between the specific incident wave and the *mean* rate of energy lost to the scattered (spike) field.

(ii) A reworking of the basic equations to describe properties of the *mean* wave established over many realizations.

(iii) Plotkin & George's (1972) perturbation scheme that specifically minimizes interpretational difficulties consequent on taking a statistical mean.

Now the remarkable fact is that all these three approaches, emphasizing mean properties to apparently different degrees, actually give rise to *identical* descriptions of the boom signature. Furthermore, it is now apparent to us that the approaches are essentially similar and describe only the stochastic mean wave, and that this mean wave gives only an irrelevant upper bound on shock thickness.

The first two schemes are described in §§3 and 4 of this paper, and are written in such a way that they lead to precisely the same equation for wave profile as that obtained by Plotkin & George. The reasons for the similarity are also given, being basically that since the turbulence is only described in the mean all the theories are thereby limited to a specification of mean effects.

We are therefore led to re-examine the issue of how, and by how much, turbulence causes wave thickening. The remaining sections of the paper deal with this question, but need some introduction to justify the method of attack. If the weak shock wave enters the turbulence as a discontinuity, then it will always remain a discontinuity. This follows from the hyperbolic nature of the governing equations, the discontinuity being a permanent feature of the characteristic which it must form (Courant & Hilbert 1962, vol. II, p. 573). The discontinuity can become weak owing to the wave energy being spread over a larger area as random turbulent convection generates an increasingly convoluted wave surface. The wave propagates (exactly) in accordance with the laws of geometrical acoustics. In principle, several weakened rays can converge onto the neighbourhood of a point, giving a seemingly continuous wave profile which is actually a succession of discontinuities. This mechanism will ultimately take over in a way that Pierce (1971) has already described, and the shock wave will appear to rise continuously over the interval separating the arrival times of the first and last ray that arrive at any point after following their individual tortuous paths through the turbulence. However it seems to us that the time required for this process is altogether too long when compared with the time available to the shock for traversing the atmospheric boundary layer in which the turbulence resides.

For many rays to converge onto a point, they must be significantly deflected from their nominal course. That is, the sharp wave front must become highly convoluted from its initial laminar shape, and that of course takes time. We can show in fact that the slope of the (still sharp) wave surface remains small during atmospheric propagation. The weak discontinuity propagates normal to its front at the speed of sound relative to the local fluid. This fluid is in turbulent motion, so that the wave acquires also this additional velocity, a velocity that includes two distinct features. First, the wave is phase shifted as it rides on the back of the turbulence. Second, the wave front is rotated, as any fluid element would be, by the rotational component of turbulent velocity. Corrsin & Karweit (1962) have shown that the angular velocity of a line element is predominantly due to the microscale turbulence and its magnitude is of the order of the root-mean-square turbulent vorticity or the ratio of the turbulence velocity u, say, to the rotational eddy scale Δ. In crossing one rotational eddy at the wave speed c, the wave is therefore rotated through an angle $(u/\Delta)(\Delta/c)$ or m, the root-mean-square Mach number of the rotational turbulence, certainly much less than 10^{-2}. In traversing the atmospheric boundary layer of scale x, the wave encounters x/Δ independent eddies, so that its accumulated angular displacement θ is $m(x/\Delta)^{\frac{1}{2}}$. Now x is about 10^3 m and Δ is probably greater than 10 cm so the angular displacement θ is very much less than unity. This estimate is identical with that made by Lighthill (1953) in considering the angular deviation of rays in turbulent flow. It also emerges again in the linear theory of the energy exchange process which we give in §5 below, the theory being based on the constraint that θ is small so that ray paths remain distinct. The depth of the ripples in the wave front can be determined from the slope and the ripple scale, which must be identical to that of the eddies that produced them, namely Δ. The ripple amplitude is therefore Δ times θ, or $m(x\Delta)^{\frac{1}{2}}$, which is at most of the order of 20 cm. The surface area is thereby increased by a factor $1+m^2x/\Delta$, which is at most about 2, so that the amplitude of the discontinuity can vary by this factor also. (The *integrated effect*, see §6, due to propagation through the *spatially variable* atmospheric boundary layer of the earth is actually very much *less* than this.) However, it cannot be eliminated; this seems to us to be a crucial deduction and leads us to say that *atmospheric turbulence is incapable of eliminating the discontinuities in a sonic boom.*

The remaining possibility of a significant turbulent thickening of the boom therefore rests on the existence of a finite rise time before the wave enters the turbulence. This is of course possible owing either to viscosity, or because the wave is already dispersed owing to non-equilibrium effects (Hodgson & Johannesen 1971). We examine this possibility in §6, where we construct an analytical scheme based on the slow linear energy transfer out of a wave packet when travelling through a slowly varying turbulent field. Phase shift effects are specifically excluded by the expedient of dealing with spectral properties. We show there that turbulence can thicken the shock only to less than about twice its initial thickness, so that here again turbulence cannot be responsible for the thousand-fold difference between the observed thickness and its Taylor value.

We are then led to the conclusion that, though turbulence is the undoubted cause of spike formation, it plays no significant part in thickening the boom structure. Furthermore, we are now aware that the shocks may well be of the fully dispersed type and that the real cause of the anomalous behaviour is to be found, as Hodgson (1972) advocates, in the non-equilibrium behaviour of air, for he has been able to demonstrate that this effect is compatible with the experimental data.

2. Equation of sound propagation through turbulence

Consider a turbulent compressible atmosphere in which c_0 denotes the speed of sound in the absence of turbulent and thermal fluctuations. When viscous diffusion effects are neglected the Navier–Stokes equation can be expressed in Lighthill's (1952) form

$$\frac{\partial^2\rho}{\partial t^2}-c_0^2\nabla^2\rho = \frac{\partial^2}{\partial x_i\partial x_j}(\rho u_i u_j)+\nabla^2(p-c_0^2\rho), \tag{2.1}$$

where ρ is the fluid density, p the pressure and u_i the fluid velocity.

The terms on the right-hand side of this equation describe three distinct physical processes. First of all they contain nonlinear components which tend to steepen acoustic wave fronts. Second, there are terms describing the *interaction* of the sound field with the inhomogeneities of the medium, i.e. with atmospheric velocity and temperature fluctuations. These are responsible for the gradual distortion and scattering of an incident wave. Finally the right side of (2.1) involves terms accounting for the *generation* of sound by the turbulent and thermal fluctuations and which function essentially independently of the incident acoustic field. We shall actually be concerned with the propagation of sound in the atmosphere at frequencies which greatly exceed those associated with this 'self-sound' of the atmospheric turbulence. It will therefore be assumed that both the turbulence and the associated aerodynamic noise fields are 'frozen' during the relatively brief time of passage of the incident sound across an eddy.

Set

$$u_i = U_i + V_i, \tag{2.2}$$

where U_i denotes the turbulent velocity fluctuations of the medium, and V_i is the velocity induced by the incident sound. Let p_0 and ρ_0 represent respectively the pressure and density in the absence of the incident sound. Then in approximating the right-hand side of (2.1) for a weakly nonlinear sound wave and in the limit of weak turbulence (i.e. $m = U/c_0 \ll 1$, where m is the root-mean-square turbulent Mach number) we retain terms which are: (i) quadratic in the turbulent fluctuations and linear in the acoustic field – representing mean *convection* effects of the medium; (ii) linear in the acoustic field *and* in the turbulent fluctuations – these *interaction* terms describe *scattering*; (iii) quadratic in the acoustic field and of zeroth order in the turbulent fluctuations – accounting for nonlinear steepening of wave fronts.

Thus the contribution of the term $\partial^2(\rho u_i u_j)/\partial x_i \partial x_j$ on the right-hand side of (2.1) remaining after subtraction of the terms which are present in the absence of the incident sound is

$$U_i U_j \frac{\partial^2 \rho'}{\partial x_i \partial x_j} + 2\rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(U_i V_j) + \rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(V_i V_j),$$

with $\rho' = \rho - \rho_0$. Similarly, if γ denotes the ratio of the specific heats of air, since variations during the passage of an incident wave may be assumed to be adiabatic, i.e. $p/p_0 = (\rho/\rho_0)^\gamma$, then in the same approximation as above, after subtraction of the steady-state terms, the contribution from the second term on the right of (2.1) is just

$$c_0^2 \frac{(\gamma-1)}{2\rho_0} \nabla^2(\rho')^2 + 2c_0^2 \nabla^2(\xi\rho'),$$

where $\xi = (c(\mathbf{x}) - c_0)/c_0$ represents the fluctuations in the speed of sound caused by the temperature inhomogeneities of the atmosphere.

Combining these results, and dropping the prime on the acoustic density perturbation, then leads to the following equation of propagation:

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2\right)\rho = U_i U_j \frac{\partial^2 \rho}{\partial x_i \partial x_j} + 2\rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(U_i V_j) + 2c_0^2 \nabla^2(\xi\rho) + \rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(V_i V_j) + \frac{c_0^2(\gamma-1)}{2\rho_0}\nabla^2(\rho^2). \tag{2.3}$$

Now Crow (1969) has analysed the relative importance of the scattering terms in this equation, and deduced that thermal effects normally constitute a small correction to the distortion and scattering associated with the turbulent velocity fluctuations. We shall therefore neglect the term involving ξ in (2.3). Further, the interaction between the sound and the turbulence described by the first term on the right-hand side of (2.3) is of second order in the turbulent velocity fluctuations. From the following analysis it will become clear that if $U_i U_j$ is replaced by its mean value (ensemble average) $\overline{U_i U_j}$, then the error involved corresponds at least to the neglect of a term which is $O(m)$ smaller. Thus we finally adopt the following equation as the starting point of the analysis of §§3 and 4:

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2\right)\rho = \overline{U_i U_j}\frac{\partial^2 \rho}{\partial x_i \partial x_j} + 2\rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(U_i V_j) + \rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(V_i V_j) + \frac{c_0^2(\gamma - 1)}{2\rho_0}\nabla^2(\rho^2). \tag{2.4}$$

3. Nonlinear theory of the mean sound field

In this and the following section equation (2.4) is used to study the propagation of the *mean* (or coherent) component of the acoustic field through a turbulent atmosphere. That is, we shall develop the analysis labelled (ii) in the introduction. The present section is devoted to the derivation of the relevant equations, then in §4 a detailed comparison of the resulting theory of sonic-boom thickening is made with the method based on Lighthill's (1953) investigation, and the recent work of Plotkin & George (1972), i.e. with approaches (i) and (iii) of the introduction.

The mean or coherent acoustic density perturbation is defined as the average of the field taken over an ensemble of realizations of the turbulent medium, and is denoted by $\bar{\rho}$. In a particular realization a correction ρ' must be applied in order to recover the actual field, so that

$$\rho = \bar{\rho} + \rho'. \tag{3.1}$$

This decomposition applies also to the velocity perturbation V_i, so that if the ensemble average of (2.4) is taken we obtain

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2\right)\bar{\rho} = \overline{U_i U_j}\frac{\partial^2 \bar{\rho}}{\partial x_i \partial x_j} + \rho_0 \frac{\partial^2}{\partial x_i \partial x_j}[\bar{V}_i \bar{V}_i + \overline{V_i' V_j'}] + \frac{c_0^2(\gamma - 1)}{2\rho_0}\nabla^2[\bar{\rho}^2 + \overline{(\rho')^2}] + 2\rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(\overline{U_i V_j'}). \tag{3.2}$$

For weak sound waves the nonlinear terms in this equation are expected to be dominated by products of the *mean field* components, which implies that the second term in *each* of the square brackets on the right of (3.2) may be neglected, giving

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2\right)\bar{\rho} = \overline{U_i U_j}\frac{\partial^2 \bar{\rho}}{\partial x_i \partial x_j} + \rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(\bar{V}_i \bar{V}_j) + \frac{c_0^2}{2\rho_0}(\gamma - 1)\nabla^2(\bar{\rho}^2) + 2\rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(\overline{U_i V_j'}). \tag{3.3}$$

The last term on the right-hand side of this equation represents the effect on the mean acoustic field of the interaction of the sound with the turbulence. To determine the nature of this interaction we proceed in the manner familiar in the theory of wave propagation in random media (see, e.g. Howe 1971*a*) and form an equation for the random component ρ' by subtracting (3.2) from the full equation (2.4):

$$\left(\frac{\partial^2}{\partial t^2}-c_0^2\nabla^2\right)\rho' = \overline{U_i U_j}\frac{\partial^2\rho'}{\partial x_i\partial x_j}+\rho_0\frac{\partial^2}{\partial x_i\partial x_j}\{2\overline{V}_i V_j'+[V_i'V_j'-\overline{V_i'V_j'}]\}$$
$$+\frac{c_0^2(\gamma-1)}{2\rho_0}\nabla^2\{2\overline{\rho}\rho'+[\rho'^2-\overline{\rho'^2}]\}$$
$$+2\rho_0\frac{\partial^2}{\partial x_i\partial x_j}\{U_i\overline{V}_j+[U_iV_j'-\overline{U_iV_j'}]\}. \tag{3.4}$$

This describes the generation of the random field ρ' by the interaction between the mean velocity field $\overline{V}_j$ and the turbulent fluctuations. The terms in square brackets represent effects of *multiple scattering* and nonlinear scrambling. When ρ' has been determined from this equation it is a simple matter to use the acoustic momentum equation in the form

$$\frac{\partial V_i'}{\partial t}+U_j\frac{\partial V_i'}{\partial x_j} = -\frac{c_0^2}{\rho_0}\frac{\partial\rho'}{\partial x_i} \tag{3.5}$$

iteratively to determine the random acoustic velocity field V_i'.

Actually reference to (3.3) reveals that we do not require a knowledge of V_i' *per se*, but only of the correlation product $\overline{U_i V_j'}$ of the random acoustic velocity and the turbulent velocity. Since V_i' is itself generated through interactions of the mean field with these velocities, it is apparent that the dominant contribution to that part of V_i' which is correlated with the turbulent velocity fluctuations at $\mathbf{x}$ *will come from those random waves which were initially scattered out of the mean field within a correlation distance l, say, of* $\mathbf{x}$. Here l denotes the correlation scale of the turbulence. Thus provided that the turbulence Mach number is sufficiently small, the multiple scattering terms on the right-hand side of (3.4) may be neglected, since these then have a minimal effect on the random waves over distances of order l. For the same reason the terms nonlinear in the random field may also be neglected. In other words, when (3.4) is to be used to determine correlations such as $\overline{U_i V_j'}$, all the terms in square brackets on the right-hand side may be dropped.

Of the remaining terms on the right-hand side of (3.4), the second and third describe interactions between the random scattered sound and the coherent field. When (3.5) is solved iteratively for V_i' and the solution used to determine the interaction term $\overline{U_i V_j'}$ of (3.3), these terms may be shown to give rise to contributions which are quadratic in the acoustic field, but $O(m^2)$ smaller than the existing quadratic terms on the right-hand side of (3.3). Similarly, the first term remaining on the right-hand side of (3.4) would give a contribution $O(m^2)$ smaller than that due to the final, *scattering* term involving $U_i\overline{V}_j$. Thus we finally arrive at the following equation:

$$\left(\frac{\partial^2}{\partial t^2}-c_0^2\nabla^2\right)\rho' = 2\rho_0\frac{\partial^2}{\partial x_i\partial x_j}(U_i\overline{V}_j), \tag{3.6}$$

which in conjunction with (3.5) is expected to yield a good approximation to the correlation product $\overline{U_i V_j'}$.

The details of the determination of this correlation product are straightforward, and follow closely the steps given in Howe (1971*b*). There it is shown that the formal substitution of the solution for the correlation product in terms of the mean field into (3.3) will result in a nonlinear equation involving the mean field alone. This equation may be expressed in the form

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2\right)\overline{\rho} = \overline{U_i U_j}\frac{\partial^2 \overline{\rho}}{\partial x_i \partial x_j} + \rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(\overline{V}_i \overline{V}_j) + \frac{c_0^2(\gamma-1)}{2\rho_0}\nabla^2(\overline{\rho}^2) + L\overline{\rho}, \tag{3.7}$$

where L is a linear operator determined by the statistical properties of the turbulence.

An explicit form for L is readily derived in the present case of waves short compared with the integral scale of the turbulence. In particular, if it is assumed that these turbulent fluctuations are isotropic we can show that

$$L = -11u^2 \frac{\partial^2}{\partial x_i^2} + \frac{2lu^2}{c_0}\frac{\partial^3}{\partial t\, \partial x_i^2} \tag{3.8}$$

(cf. Howe 1971*b*, §7). In this result $u^2 = \frac{1}{3}U_i U_i$ and the integral scale l is defined by

$$l = \frac{2}{\pi u^2}\int_0^\infty \frac{E(\kappa)}{\kappa}\, d\kappa, \tag{3.9}$$

where $E(\kappa)$ is the energy spectrum of the turbulent fluctuations (Batchelor 1953, p. 36), i.e.

$$\tfrac{3}{2}u^2 = \int_0^\infty E(\kappa)\, d\kappa.$$

Thus on substituting into (3.7) we deduce that the mean field satisfies

$$\frac{\partial^2 \overline{\rho}}{\partial t^2} - c_0^2[1 - 10m^2]\nabla^2 \overline{\rho} = \frac{2lu^2}{c_0}\frac{\partial^3 \overline{\rho}}{\partial t\, \partial x_i^2} + \rho_0 \frac{\partial^2}{\partial x_i \partial x_j}(\overline{V}_i \overline{V}_j) + \frac{c_0^2(\gamma-1)}{2\rho_0}\nabla^2(\overline{\rho}^2). \tag{3.10}$$

The effect of the random inhomogeneities on the mean field is twofold. The first term on the right-hand side of (3.8) corresponds to a reduction in the phase velocity below that for propagation in free space, and arises because the inhomogeneities compel the sound to propagate along irregular paths with a corresponding increase in the travel time between two fixed points. The second term is a third-order derivative and causes the mean field to decay. This decay represents the natural compensation in the energy balance as the energy content of the random, scattered field increases.

4. Application of the nonlinear mean wave equation to sonic-boom propagation

We now proceed to examine the relationship between methods (i), (ii) and (iii) of the introduction for the determination of shock thickness. We shall do this by first examining the consequences of applying the mean wave equation

(3.10) to the problem of sonic-boom propagation (method (ii)) and then relating the result to an analysis based on Lighthill's theory (i) and the prediction of Plotkin & George (iii).

We shall regard the mean profile of the sonic boom as a weak Taylor shock (Lighthill 1956) with a pressure jump $\Delta\overline{p}_0$. If the shock is plane and propagates in the positive-x direction (3.10) reduces to

$$\frac{\partial^2\overline{\rho}}{\partial t^2}-c_0^2(1-10m^2)\frac{\partial^2\overline{\rho}}{\partial x^2}=\frac{2lu^2}{c_0}\frac{\partial^3\overline{\rho}}{\partial t\,\partial x^2}+\rho_0\frac{\partial^2}{\partial x^2}(\overline{V})^2+\frac{c_0^2(\gamma-1)}{2\rho_0}\frac{\partial^2}{\partial x^2}(\overline{\rho})^2, \tag{4.1}$$

where $\overline{V}$ is the x component of $\overline{V}_j$.

Now for such a wave we have

$$\frac{\partial^2}{\partial t^2}-c_0^2\frac{\partial^2}{\partial x^2}\simeq-2c_0\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+c_0\frac{\partial}{\partial x}\right). \tag{4.2}$$

Making this substitution in (4.1) and integrating with respect to x then gives

$$\frac{\partial\overline{\rho}}{\partial t}+c_0(1-5m^2)\frac{\partial\overline{\rho}}{\partial x}+\frac{\rho_0}{2c_0}\frac{\partial}{\partial x}(\overline{V})^2+\frac{c_0(\gamma-1)}{4\rho_0}\frac{\partial}{\partial x}(\overline{\rho})^2=-\frac{lu^2}{c_0^2}\frac{\partial^2\overline{\rho}}{\partial x\,\partial t}\simeq\frac{lu^2}{c_0}\frac{\partial^2\overline{\rho}}{\partial x^2}, \tag{4.3}$$

the constant of integration vanishing identically because gradients vanish at $x=\pm\infty$. If the approximate relation $\overline{\rho}/\rho_0=\overline{V}/c_0$ is now used to eliminate $\overline{V}$ in the nonlinear term, then we have

$$\frac{\partial\overline{\rho}}{\partial t}+c_0(1-5m^2)\frac{\partial\overline{\rho}}{\partial x}+\frac{c_0(\gamma+1)}{2\rho_0}\overline{\rho}\frac{\partial\overline{\rho}}{\partial x}=\frac{lu^2}{c_0}\frac{\partial^2\overline{\rho}}{\partial x^2}. \tag{4.4}$$

Alternatively this may be expressed in terms of the pressure perturbation $\overline{p}$, which is related to $\overline{\rho}$ by $\overline{p}/\overline{\rho}=c_0^2$:

$$\frac{\partial\overline{p}}{\partial t}+c_0(1-5m^2)\frac{\partial\overline{p}}{\partial x}+\frac{(\gamma+1)\,c_0}{2\gamma p_0}\overline{p}\frac{\partial\overline{p}}{\partial x}=\frac{lu^2}{c_0}\frac{\partial^2\overline{p}}{\partial x^2}. \tag{4.5}$$

This is the desired equation for the mean or coherent pressure field associated with a plane shock wave propagating through turbulence. Neglecting the small, and physically insignificant, correction to the free-space propagation velocity we have to a good approximation

$$\frac{\partial\overline{p}}{\partial t}+c_0\frac{\partial\overline{p}}{\partial x}+\frac{(\gamma+1)\,c_0}{2\gamma p_0}\overline{p}\frac{\partial\overline{p}}{\partial x}=\frac{lu^2}{c_0}\frac{\partial^2\overline{p}}{\partial x^2}. \tag{4.6}$$

This will be recognized as Burger's equation, familiar in the Taylor theory of weak shock waves. That theory involves an identical equation except that here the *dissipation* term on the right-hand side of (4.6) depends on the properties of the turbulent scatterers rather than the viscosity of the fluid. A steady shock thickness is obtained by balancing the diffusive spreading of the wave produced by this term against nonlinear steepening.

It is an easy matter to solve (4.6) for a monotonic weak shock profile. The method of solution is well known and is discussed in considerable detail by Lighthill (1956) for the case of shocks controlled by viscous dissipation. Following precisely Lighthill's analysis we deduce a steady-state 'thickness' δ of the mean shock wave based on the maximum slope:

$$\delta = 16\gamma p_0 l m^2/(\gamma+1)\,\Delta\bar{p}_0. \tag{4.7}$$

We must be careful not to confuse this thickness of the *mean shock* with the *mean thickness* of the actual shock. When the shock propagates through the turbulent atmosphere it is naturally propagating through a medium whose typical scales of variation are very much larger than the thickness of the shock. This implies that it should propagate according to the laws of geometrical acoustics. Consequently the actual position of the shock executes a random walk caused by the convective turbulent velocity fluctuations about the position the shock would have attained had it been propagating in free space.

Now the rigorous theory of geometrical acoustics (Courant & Hilbert 1962, p. 573) predicts that an initially discontinuous wave front will remain discontinuous. In spite of this an abrupt pressure jump averaged over the random walk gives an apparently thick mean wave front. Analytically this 'thick' average is caused by the diffusive nature of the *scattering* term on the right-hand side of (4.6) and casts serious doubts on the ability of that equation to describe in even a remotely approximate manner the properties of an individual realization of the wave field.

Let us now observe that when the nonlinear term in (4.6) is neglected, a harmonic wave of the form $\bar{p} = \exp\{i(kx-\omega t)\}$ would satisfy

$$\left.\begin{aligned} \omega &= c_0 k - ilu^2k^2/c_0, \\ \bar{p} &= \exp\{ik(x-c_0t) - lu^2k^2t/c_0\}. \end{aligned}\right\} \tag{4.8}$$

Since the intensity I, say, of the mean wave is proportional to $\bar{p}^2$, this implies that

$$\frac{dI}{dt} = -\frac{2lu^2}{c_0}k^2 I, \tag{4.9}$$

the differentiation following the motion of the wave at the speed of sound. This agrees exactly with the result obtained by Lighthill (1953) in his treatment of turbulent scattering of high frequency sound, and as such apparently constitutes a basis for determining shock thickening (method (i) of the introduction). To do this one merely inverts the argument leading to (4.9). That equation may be taken to imply the existence of a small negative imaginary component in the frequency ω of a harmonic wave, as in (4.8), accounting for the slow exponential decay of the wave due to scattering. This may then be incorporated into the Navier–Stokes equation (in which turbulent velocity fluctuations are omitted) by the simple formal expedient of replacing $\partial/\partial t$ by $\partial/\partial t - (lu^2/c_0)\,\partial^2/\partial x^2$, a procedure which would then lead to precisely equation (4.6), and to a shock thickness identical with (4.7).

However, Lighthill warned against interpreting the dissipation implied by (4.9) as a genuine energy loss because the result is derived only in the *stochastic* mean, and is probably only approximated to in a single realization.

Consider next the determination of shock thickening undertaken by Plotkin & George (1972; method (iii) of the introduction). By means of a double parameter expansion procedure based on the shock strength and the root-mean-square turbulence Mach number, Plotkin & George arrive at essentially the following equation for the pressure perturbation:

$$\frac{\partial P}{\partial t}+c_0\frac{\partial P}{\partial x}+\frac{(\gamma+1)\,c_0}{2\gamma p_0}P\frac{\partial P}{\partial x}=\mathscr{D}P. \tag{4.10}$$

Actually P denotes the pressure perturbation from which the first scattered 'spikes' have been excluded, so that it hopefully represents a smoothed-out (but not averaged) wave profile. On the right of (4.10) $\mathscr{D}$ denotes a linear operator which accounts for the interactions of this pressure field with the turbulence, and which therefore depends on spatial location and the particular realization of the turbulence. Because it is generally not possible to determine $\mathscr{D}$ precisely, but only its statistical properties, Plotkin & George argue that provided a solution is sought only over distances which are large compared with the integral scale l of the turbulence, it is permissible to replace $\mathscr{D}$ by its ensemble average $\langle\mathscr{D}\rangle$, say. In this way they deduce that

$$\mathscr{D}\simeq\langle\mathscr{D}\rangle\simeq(lu^2/c_0)\,\partial^2/\partial x^2, \tag{4.11}$$

and so arrive at precisely equation (4.6) for the propagation of the mean wave and thence to the prediction (4.7) for the steady-state shock thickness.

To understand why Plotkin & George should obtain precisely the results derived from the mean wave equation, it is interesting to note that to some extent their analysis was anticipated by Lighthill. In the appendix of his (1953) paper Lighthill determines the second scattered acoustic field and shows that it is correlated with the incident wave in such a manner that it just compensates for the energy loss from the incident field *to* the first scattered waves. Thus in calculating the mean rate of energy dissipation of P, the *wave field less the first scattered 'spikes'*, one obtains precisely the mean effect on the incident wave – a second-order effect – coinciding exactly with the mean wave analysis, since the 'spike' field is *actually* dominated by the *random walk* anomaly.

We conclude therefore that there is a basic identity between the apparently distinct methods (i), (ii) and (iii) of the introduction. It has been argued that the turbulent-induced random walk of the wave front dominates the scattered field in the high frequency limit associated with a shock wave (strong *forward scatter*); in other words in an individual realization practically *all* of the scattered sound actually propagates in the *same* direction and at the *same* speed as the incident wave. This implies that in these circumstances the mean or coherent wave can in no way approximate to the properties of a particular realization of the field; the energy content of the mean wave certainly decays according to Lighthill's formula (4.9), but *because* it relates only to the mean wave it should be regarded essentially as the decay of the *coherence* of the wave front and not of the energy content of the actual wave.

5. Linear theory of acoustic energy exchange processes

From the material of the preceding sections we conclude that it is highly unreasonable to regard the thickness of the mean shock wave as being truly representative of the thickness of individual realizations of a shock propagating through turbulence, and that calculations based on methods (i), (ii) and (iii) of the introduction result in equally fallacious predictions of that thickness. Actually (4.7) shows that the mean wave thickness depends critically on the size of the integral scale of the turbulence, which in the atmospheric boundary layer is of the order of 100 metres or more. Most investigators will instinctively feel that coarse grain features of turbulent motions, which determine the integral scale l, cannot possibly participate in what is after all a delicate balance between local nonlinearities and dissipative mechanisms. Even if large-scale features were responsible for some of the attenuation, it can reasonably be argued that smaller scale motions would be more efficient in affecting the wave, if only because there would then be a very much smaller mismatch between interacting length scales.

Now in an examination of single-scattering theory applied to an ideal shock wave propagating through turbulence Crow (1969) was able to separate the first scattered field into two distinct components. The first was interpreted as a *phase shift* which accounts for the local variable velocity of propagation of the incident wave caused by turbulent convection. The second component was much smaller in magnitude and was regarded as the genuinely scattered sound. It appears, therefore, that a rational theory of sonic-boom thickening by turbulence must be capable of taking account of these physically distinct contributions to the scattered field. Plotkin & George (1972) recognized this difficulty, but were unable to formulate a mathematically tractable programme to extract the true intensity of the scattered sound.

In this section we examine the propagation of short sound waves through turbulence in terms of a theory of multiple scattering proposed in a companion paper (Howe 1973), with a view to effecting this separation. In that paper an integro-differential *kinetic* equation is derived for the mean-square Fourier coefficients of waves propagating in inviscid turbulence. If at position $\mathbf{X}$ and time T the acoustic energy in the wavenumber range $(\mathbf{k}, d\mathbf{k})$ is denoted by $\mathscr{E}(\mathbf{k}, \mathbf{X}, T)\, d\mathbf{k}$ per unit volume, then Howe shows that asymptotically as the turbulence Mach number m tends to zero $\mathscr{E}(\mathbf{k}, \mathbf{X}, T)$ satisfies

$$\frac{\partial \mathscr{E}(\mathbf{k})}{\partial T} + \frac{c_0 \mathbf{k}}{k} \cdot \frac{\partial \mathscr{E}(\mathbf{k})}{\partial \mathbf{X}} = \frac{4\pi}{c_0 k^3} \int (\mathbf{k} \,.\, \mathbf{K})^2 k_i k_j \Phi_{ij}(\mathbf{k} - \mathbf{K}) \times [\mathscr{E}(\mathbf{K}) - \mathscr{E}(\mathbf{k})]\, \delta(K^2 - k^2)\, d\mathbf{K}, \tag{5.1}$$

where $\Phi_{ij}(\mathbf{k})$ is the *spectrum* of the turbulent velocity fluctuations, which is defined as the Fourier transform of the spatial velocity correlation tensor (Batchelor 1953, p. 26). Since the spectrum function is proportional to u^2 equation (5.1) can be shown to imply that changes in $\mathscr{E}(\mathbf{k}, \mathbf{X}, T)$ due to scattering occur over distances and times which are of order $1/m^2$, i.e. over distances which are generally large when compared with the extent of the wave form. It is in this sense that $\mathscr{E}(\mathbf{k}, \mathbf{X}, T)$ can be considered to depend independently on the apparently conjugate variables $\mathbf{k}$ and $\mathbf{X}$. That is, the space and time variables $\mathbf{X}$ and T are really 'slow' variables, and we may regard the Fourier representation of the wave field in terms of the wavenumber vectors $\mathbf{k}$ as being valid locally inside boxes of dimension $\Delta\mathbf{X}$, say, which is small compared with the distances

$\mathbf{X}$ ($\sim (1/m^2)$) over which significant changes occur in the amplitudes of the wave components, yet large compared with a typical wavelength of the field.

To interpret the significance of the term on the right-hand side of (5.1) observe that the integration with respect to $\mathbf{K}$ extends over the whole spectrum of acoustic waves. That part of the integrand associated with $\mathscr{E}(\mathbf{K})$ in the square brackets contributes a net *influx* of energy into the element $(\mathbf{k}, d\mathbf{k})$ of wavenumber space due to scattering from all other waves of energy $\mathscr{E}(\mathbf{K})\,d\mathbf{K}$ per unit volume. Similarly the second term in the square brackets gives the rate of *loss* of energy from $(\mathbf{k}, d\mathbf{k})$ because of scattering *into* all possible other wave modes $(\mathbf{K}, d\mathbf{K})$. Note that at $\mathbf{K} = \mathbf{k}$ the term in square brackets vanishes identically, i.e. there is no net gain or loss of energy, and corresponds to the elimination of *phase shift* from the energy balance. Finally, the presence of the delta function implies that, since, as before, the turbulence is assumed to be *frozen*, interacting acoustic waves must have the same frequency.

Now when the acoustic wavelength is much smaller than the correlation scale l of the turbulence, the main contribution to the integral in (5.1) is from the region near $\mathbf{K} = \mathbf{k}$ (cf. Lighthill 1953). Hence that integral may be evaluated approximately by expanding $\mathscr{E}(\mathbf{K})$ about $\mathbf{K} = \mathbf{k}$. Carrying the expansion to second order then reduces the integro-differential equation to the following *diffusion equation* for the distribution of acoustic energy in wavenumber space:

$$\frac{\partial \mathscr{E}(\mathbf{k})}{\partial T} + c_0 \frac{\partial \mathscr{E}(\mathbf{k})}{\partial X_{\parallel}} = \frac{c_0 m^2 k^2}{2\Delta} \nabla_{\perp}^2 \mathscr{E}(\mathbf{k}). \tag{5.2}$$

In this equation the following notation has been adopted: $X_{\parallel}$ is the space coordinate parallel to the wavenumber vector $\mathbf{k}$; $\nabla_{\perp}^2$ denotes the two-dimensional Laplacian operator in wavenumber space in the plane normal to $\mathbf{k}$. The *length* Δ is given in terms of the energy spectrum $E(\kappa)$ of the turbulence by

$$\frac{1}{\Delta} = \frac{\pi}{2u^2} \int_0^{\infty} \kappa E(\kappa)\, d\kappa, \tag{5.3}$$

and is of the same order of magnitude as the Taylor microscale (Batchelor 1953, p. 47). Actually it has been implicitly assumed in deriving (5.3) that the turbulence is locally isotropic. However, it will be clear that Δ is really determined by those wavenumber components of the turbulence which are large compared with those of the energy-containing eddies; in fact significant contributions to the integral come only from eddies within the inertial subrange and the viscous dissipation range. These are usually thought to be essentially isotropic, so that the approximation leading to (5.3) is perhaps not unduly restrictive.

The quantity which will be of significance in the ensuing discussion is the *scattering diffusivity* μ', defined by

$$\mu' = 2m^2/\Delta. \tag{5.4}$$

In the practical problem of sonic-boom propagation both m and Δ will vary with position in the atmosphere, but over distances which may be assumed large compared with the spatial extent of the wave form.

Now (cf. Howe 1973, §7) using relations given in Batchelor's (1953) monograph, it is an easy matter to deduce that the integral in (5.3) can be alternatively expressed as an integral involving the longitudinal correlation function $f(r)$, viz.

$$\frac{1}{\Delta} = -4\int_0^\infty \frac{1}{r}\frac{\partial f}{\partial r}\,dr. \tag{5.5}$$

Also, if $D(r) = \overline{[u(r_0)-u(r_0+r)]^2}$ is the longitudinal *structure function*, then it is known that for distances within the inertial subrange

$$D \simeq K\epsilon^{\frac{2}{3}}r^{\frac{2}{3}} \tag{5.6}$$

(Tatarski 1961, pp. 27–58), where ϵ is the mean rate of dissipation of turbulent kinetic energy by viscous stress and K is a dimensionless universal constant $\simeq 1{\cdot}9$ (Crow 1969). For smaller values of the separation distance r, smaller than the Kolmogorov scale $(\nu^3/\epsilon)^{\frac{1}{4}}$, ν being the kinematic viscosity, we have instead

$$D \simeq \epsilon r^2/15\nu. \tag{5.7}$$

There is no theoretical link between the forms (5.6) and (5.7) of the structure function at intermediate separations, so that, following Crow (1969), we formally adopt the interpolation formula

$$D = \alpha r^2/(1+\beta r^2)^{\frac{2}{3}}, \tag{5.8}$$

where
$$\alpha = \epsilon/15\nu, \quad \beta = \epsilon^{\frac{1}{2}}/(15\nu K)^{\frac{3}{2}}. \tag{5.9}$$

The actual form of the structure function would not be expected to deviate significantly from (5.8).

If we now observe that

$$\frac{\partial f}{\partial r} = -\frac{1}{2u^2}\frac{\partial D}{\partial r}, \tag{5.10}$$

then the integral expression (5.5) for Δ becomes

$$\frac{1}{\Delta} = \frac{2}{u^2}\int_0^\infty \frac{1}{r}\frac{\partial D}{\partial r}\,.\,dr, \tag{5.11}$$

and using the formula (5.8) we deduce that

$$\frac{1}{\Delta} = \frac{8\alpha\,\Gamma(\frac{3}{2})\,\Gamma(\frac{7}{6})}{u^2\beta^{\frac{1}{2}}\,\Gamma(\frac{5}{3})}. \tag{5.12}$$

The convergence of the integral (5.11) for the interpolation formula (5.8) may generally be regarded as confirming the conjecture that Δ is determined essentially by the eddies contained within and beyond the inertial subrange. The recent measurements of u and ϵ undertaken in the lower atmosphere by Sheih, Tennekes & Lumley (1971) indicate that Δ is typically of the order of 10–15 cm.

Consider next the following initial-value problem associated with the diffusion equation (5.2). At time $T = 0$ a plane wave enters a region of turbulence at $X = 0$ which extends from $X = 0$ to $X = \infty$, and proceeds to propagate in the positive-X direction. Let $\mathbf{k}_0 = (k_0, 0, 0)$ denote the incident wavenumber, then with respect to a suitable system of units, we may set

$$\mathscr{E}(\mathbf{k}) = \delta(\mathbf{k}_x - k_0)\,\delta(\mathbf{k}_\perp)\,\delta(X) \tag{5.13}$$

at $T = 0$, where $\mathbf{k}_\perp$ is the wavenumber component perpendicular to the x axis. In (5.13) the function $\delta(X)$ merely implies that the extent of the wave packet is small compared with the distance over which significant changes in the properties of the wave profile occur, in other words, that the spatial extent of the wave may be regarded as contained entirely within a 'box' of dimension $\Delta\mathbf{X}$.

The full solution of (5.2) subject to this initial condition is difficult because of the dependence of the derivatives on the direction of propagation (i.e. on $\mathbf{k}$). However, for sufficiently small propagation distances an approximate solution can be obtained on the assumption that the angular divergence of the wave packet caused by scattering as it propagates through the turbulence is small, so that, in particular, the derivative $\partial/\partial X_\parallel$ may be approximated by $\partial/\partial X$. The appropriate solution is then easily derived by Fourier analysis and may be expressed in the form

$$\mathscr{E}(\mathbf{k}) = \frac{\delta(k-k_0)\,\delta(X-c_0 T)}{k^2\pi\mu(X)}\exp\left\{\frac{-\theta^2}{\mu(X)}\right\}, \tag{5.14}$$

where the angular divergence θ is measured from the positive-X direction to the direction of propagation $\mathbf{k}$.

The dimensionless quantity $\mu(X)$ is the *integrated scattering diffusivity*:

$$\mu(X) = 2\int_0^X \frac{m(\xi)^2}{\Delta(\xi)}\,d\xi, \tag{5.15}$$

the integral being along the path of propagation of the wave. In the atmospheric boundary layer the scattering diffusivity $\mu' = 2m^2/\Delta$ generally varies with position, but as mentioned above, the length scale associated with its variation is usually large compared with the linear extent of the wave profiles to be considered below. Note that the validity of the solution (5.14) requires that the integrated scattering diffusivity $\mu(X)$ be small.

It is a relatively simple matter to extend this solution to cover more general initial wave energy spectra. Suppose that at $T = 0$

$$\mathscr{E}(\mathbf{k}) = G(k_x)\,\delta(\mathbf{k}_\perp)\,\delta(X), \tag{5.16}$$

where, without loss of generality, and for waves propagating initially in the positive-X direction, it may be assumed that $G(k_x)$ is non-zero only for $k_x > 0$. Then the subsequent development of the spectrum due to propagation through the turbulence may be derived by convoluting (5.16) with the elementary solution (5.14). In this manner we find that for $T > 0$

$$\mathscr{E}(\mathbf{k}) = \frac{G(k)\,\delta(X-c_0 T)}{k^2\pi\mu(X)}\exp\left\{\frac{-\theta^2}{\mu(X)}\right\}. \tag{5.17}$$

This result, illustrating the lateral divergence of the acoustic wave, will be used in the next section to study sonic-boom thickening.

6. Linear theory of turbulent shock thickening

In order to avoid the difficulties associated with the analyses of shock thickness discussed in §§3 and 4, we now propose an alternative definition of thickness which specifically excludes spurious contributions due to possible phase shift effects.

Let $p(x,y,z,t)$ denote the pressure perturbation associated with a shock propagating nominally in the positive-x direction. Then if the shock has strength $\Delta\bar{p}_0$, we define its thickness δ by

$$\frac{1}{\delta} = \overline{\frac{1}{(\Delta\bar{p}_0)^2}\int_{-\infty}^{\infty}\left[\frac{\partial p}{\partial x}(x,y,z,t)\right]^2 dx} = \frac{1}{(\Delta\bar{p}_0)^2}\int_{-\infty}^{\infty}\overline{\left(\frac{\partial p}{\partial x}\right)^2} dx, \tag{6.1}$$

where the overbar denotes an average taken over an ensemble of realizations of the turbulent field.

To see that this is a reasonable definition note first that, because it involves an integration in the nominal direction of propagation, the exact location of the shock is not important, i.e. all effects of phase shift are automatically excluded.

It may be argued, however, that an unconscionably large contribution to the integral in (6.1) comes not only from the wave front but also from the presence of spikey irregularities of the wave profile caused by propagation through the turbulence (see figures 1(a), 2(a) and 3(a)). Actually this cannot be the case since *on average* the effect of the turbulence *must* be to smooth out any irregularities, and initially the only irregularity is that due to the abrupt pressure rise at the wave front. Alternatively, observe that

$$\overline{\left(\frac{\partial p}{\partial x}(x,y,z,t)\right)^2} = \lim_{\xi,\zeta\to 0}\frac{\partial^2}{\partial\xi\,\partial\zeta}I(\mathbf{x},t,\xi,\zeta), \tag{6.2}$$

where
$$I(\mathbf{x},t,\xi,\zeta) = \overline{p(x+\xi,y,z,t)\,p(x+\zeta,y,z,t)}. \tag{6.3}$$

The function $I(\mathbf{x},t,\xi,\zeta)$ may be regarded as a correlation coefficient relating points on the wave profile a distance $|\xi-\zeta|$ apart, and is necessarily *smooth*, in the sense that the spikey irregularities, although contributing on an average to the form of the function, do not involve it in violent fluctuations. When ξ and ζ are both small continuity ensures that $I(\mathbf{x},t,\xi,\zeta)$ approximates to the mean-square pressure perturbation distribution. That distribution is clearly smooth and effectively uniform except in the vicinity of the wave front where there is a rapid but smooth rise in the overpressure from zero. In fact figure 1(b), for example, shows that considerable shock thickening can occur without the presence of a significant field of spikey irregularities.

The definition (6.1) involves the co-ordinates y and z transverse to the nominal wave front; they may be eliminated by averaging over an area $\Delta Y\Delta Z$ of the shock front, where, in accordance with the discussion of §5, ΔY and ΔZ are large compared with the characteristic 'ripples' of the shock front. Of course, if the turbulence were homogeneous and isotropic then the co-ordinates y and z would *not* appear in the integral of (6.1).

Further, since $\overline{(\partial p/\partial x)^2}$ tends rapidly to zero away from the shock front, the range of integration in (6.1) may be restricted to a region of the x axis of length ΔX, say, within which the integrand is non-zero. Thus we may now set

$$\frac{1}{\delta} = \frac{\int_{\Delta \mathbf{X}} \overline{\left(\frac{\partial p}{\partial x}\right)^2}\, d\mathbf{x}}{\Delta Y \Delta Z\, (\Delta \overline{p}_0)^2}, \tag{6.4}$$

where the integration is taken over a 'box' of volume $\Delta \mathbf{X} = \Delta X \Delta Y \Delta Z$ containing that portion of the wave profile in which $\overline{(\partial p/\partial x)^2}$ is non-zero.

To determine the value of this integral express the wave profile in the form of a Fourier expansion

$$p(\mathbf{x}) = \sum_n a(\mathbf{k}_n, t)\, e^{i\mathbf{k}_n \cdot \mathbf{x}} \tag{6.5}$$

valid within the box $\Delta \mathbf{X}$, so that

$$\partial p/\partial x = \sum_n i k_{nx} a(\mathbf{k}_n, t)\, e^{i\mathbf{k}_n \cdot \mathbf{x}}. \tag{6.6}$$

Thus squaring and integrating with respect to $\mathbf{x}$ over the volume of the box and taking the ensemble average yields

$$\int_{\Delta \mathbf{X}} \overline{\left(\frac{\partial p}{\partial x}\right)^2}\, d\mathbf{x} = \sum_n k_{nx}^2\, \overline{|a(\mathbf{k}_n, t)|^2}\, \Delta \mathbf{X}, \tag{6.7}$$

since the functions $\exp(i\mathbf{k}_n \,.\, \mathbf{x})$ are mutually orthogonal within the box.

But, $\overline{|a(\mathbf{k}_n, t)|^2}/\Delta \mathbf{k}$ is proportional to the mean energy per unit volume of the spectral component of wavenumber $\mathbf{k}_n$, which may be denoted by $\mathscr{E}(\mathbf{k}_n)$, say. Suppose that the shock wave enters the turbulence at time $T = 0$, then initially we may set

$$\overline{|a(\mathbf{k}, 0)|^2} = G(\mathbf{k}_x)\, \delta(\mathbf{k}_\perp)\, \delta(X)\, \Delta \mathbf{k} \tag{6.8}$$

and the evolution of the energy spectrum can be determined by the method of the previous section.

Thus we have

$$\int_{\Delta \mathbf{X}} \overline{\left(\frac{\partial p}{\partial x}\right)^2}\, d\mathbf{x} = \int k_x^2 \mathscr{E}(\mathbf{k})\, \Delta \mathbf{X} d\mathbf{k}. \tag{6.9}$$

Using the solution (5.17), we have, correct to the order of approximation implied by that solution,

$$\int_{\Delta \mathbf{X}} \overline{\left(\frac{\partial p}{\partial x}\right)^2}\, d\mathbf{x} = \frac{\delta(X - c_0 T)\, \Delta X \Delta Y \Delta Z}{\pi \mu(X)} \times \int_0^\infty \theta\, d\theta \int_0^{2\pi} d\phi \int_0^\infty k^2\, G(k)\, (1-\theta^2)\, e^{-\theta^2/\mu(X)}\, dk. \tag{6.10}$$

Now the delta function on the right specifies the location of the box $\Delta \mathbf{X}$ under consideration, so that $\delta(X - c_0 T)\, \Delta X = 1$ for that box. Hence carrying out the integrations gives

$$\int_{\Delta \mathbf{X}} \overline{\left(\frac{\partial p}{\partial x}\right)^2}\, d\mathbf{x} = \Delta Y \Delta Z\, [1 - \mu(X)] \int_0^\infty k^2 G(k)\, dk. \tag{6.11}$$

Substituting into (6.4) we obtain

$$\frac{1}{\delta} = \frac{[1 - \mu(X)]}{(\Delta \overline{p}_0)^2} \int_0^\infty k^2 G(k)\, dk. \tag{6.12}$$

However if δ_0 denotes the initial thickness of the shock on entering the turbulence, then $\delta_0 = (\Delta \bar{p}_0)^2 \Big/ \int_0^\infty k^2 G(k)\,dk$, so that, for small $\mu(X)$, we finally have

$$\delta = \delta_0[1+\mu(X)]. \tag{6.13}$$

In particular, if $\delta_0 = 0$, this implies that, at least over distances for which the approximate solution (5.17) remains valid, the shock thickness is vanishingly small.

Let us now apply this result to sonic-boom propagation through the atmospheric boundary layer. The turbulent motions of the lower atmosphere generally extend to a height of about 10^5 cm. Following Crow (1969) we shall suppose that the atmospheric boundary layer behaves approximately like a wind-tunnel boundary layer under a uniform free stream. This permits the turbulence dissipation function ϵ to be expressed in a universal form

$$\epsilon = u_*^3 \delta_*^{-1} W(y/\delta_*), \tag{6.14}$$

where u_* is the *friction velocity* and is equal to the square root of the ratio of the surface stress to the air density. The altitude δ_* is the height at which the wind speed is 99·5 % of its free-stream value; y is the distance measured vertically from the ground, and $W(y/\delta_*)$ is a universal function known from wind-tunnel data.

Now from the definition (5.15) and the expressions (5.9) and (5.12) we have

$$\mu(X) = \frac{16\Gamma(\frac{3}{2})\,\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{3})} \int_0^X \frac{m(\xi)^2}{u(\xi)^2} \left(\frac{\epsilon^3 \kappa^3}{15\nu}\right)^{\frac{1}{4}} d\xi, \tag{6.15}$$

the integration being from $\xi = 0$, where the shock enters the boundary layer, to $\xi = X$. It is convenient to express this result as an integral over the vertical height y, and to extend the path of integration down to ground level in order to take account of the full effect of the atmospheric boundary layer. For an aircraft moving supersonically at Mach number M, the appropriate transformation for an observer immediately below the flight path is $d\xi = -M\,dy/(M^2-1)^{\frac{1}{2}}$, so that if $\lambda = y/\delta_*$ then

$$\mu_{y=0} = \mu^* = \frac{16\Gamma(\frac{3}{2})\,\Gamma(\frac{7}{6})\,M}{\Gamma(\frac{5}{3})\,(M^2-1)^{\frac{1}{2}}} \left(\frac{K^3}{15}\right)^{\frac{1}{4}} m_*^{\frac{2}{4}} \left(\frac{u_*\delta_*}{\nu}\right)^{\frac{1}{4}} \int_0^1 W(\lambda)^{\frac{3}{4}}\,d\lambda, \tag{6.16}$$

where $m_* = u_*/c_0$.

Now using data taken from Bradshaw, Ferris & Atwell (1967) for the case of a wind-driven boundary layer under zero pressure gradient, we can estimate that

$$\int_0^1 W(\lambda)^{\frac{3}{4}}\,d\lambda \simeq 9{\cdot}5. \tag{6.17}$$

Hence, by evaluating the remaining constant terms in (6.16) we finally deduce that

$$\mu_* \simeq 114 \frac{m_*^{\frac{2}{4}} M}{(M^2-1)^{\frac{1}{2}}} \left(\frac{u_*\delta_*}{\nu}\right)^{\frac{1}{4}}. \tag{6.18}$$

It is now clear that in all cases of practical interest the integrated scattering diffusivity μ^* is small. Typically the friction velocity $u_* \simeq 30$–$100\ \mathrm{cm\,s^{-1}}$ (Lumley

& Panofsky 1964, p. 100; Sheih, Tennekes & Lumley 1971), and rarely exceeds $150\,\mathrm{cm\,s^{-1}}$. The boundary-layer thickness $\delta_* \simeq 10^5\,\mathrm{cm}$ and $\nu = 0{\cdot}15\,\mathrm{cm^2\,s^{-1}}$. Thus for a supersonic aircraft such as *Concorde* μ^* is typically of order 10^{-2}–10^{-1} with a maximum value of about 0·3.

Hence the formula (6.13) derived above implies that propagation through the turbulent atmospheric boundary layer apparently produces at most an increase of about 30 % in the sonic-boom rise time!

7. Conclusions

We have shown that the two departures in practical measurements of the sonic boom from ideal quiescent flow theory have essentially different causes. Namely, though the spike structure is generated by an interaction of the boom with the turbulence it encounters in passing from the aircraft to the ground the anomalous shock thickening cannot be caused by turbulence.

The earlier work suggesting turbulence to be the cause of wave thickening is shown to describe only properties of a wave established in the stochastic mean, and presents an irrelevant upper bound on wave thickness. We have then shown that the distance travelled by the boom through turbulence is too small to cause *any* turbulent thickening of an initially discontinuous pressure signal. But turbulence can thicken a boom with initial finite rise time by, at most, a factor of about 2. Turbulence cannot therefore be the cause of the one thousandfold discrepancy between practical measurements of boom thickness and the Taylor value.

In view of the now known tendency for weak shocks to attain a fully dispersed profile owing to non-equilibrium gas effects, and the demonstration that these effects are consistent with practical measurements (Hodgson 1972), there seems no remaining mystery regarding the boom's 'anomalous' structure.

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